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A new proof of Mayer's theorem ☆

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Abstract

This paper gives a new proof of Mayer's theorem concerning the convergence of powers of an interval matrix. © 2002 Elsevier Science Inc. All rights reserved.

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1. Introduction

In 1984, Mayer [2] studied the conditions for the sequence $\{\mathscr{A}^l : l \in \mathbb{N}\}$ of the powers of an $n \times n$ interval matrix $\mathscr{A}$ to converge to the null matrix 0. The remarkable result proposed by Mayer states that $\mathscr{A}^l \to 0$ as $l \to \infty$ if and only if $\rho(\tilde{\mathscr{A}}) < 1$, where $\tilde{\mathscr{A}}$ is a point matrix constructed by the $(*)$ property. Mayer established this theorem by using the Perron–Frobenius theorem. In this paper, we give an alternative proof using decomposition method. For easy reference, we follow the notations in [2].

2. A new proof for Mayer's theorem

Let $\mathscr{A} = (A_{ij})$ be an $n \times n$ (real) interval matrix, that is, each A_{ij} is a real compact interval $[\underline{a}_{ij}, \overline{a}_{ij}]$, where $\underline{a}_{ij} \leqslant \overline{a}_{ij}$. $\mathscr{A}_j$ denotes the jth column of $\mathscr{A}$. For

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$\mathscr{A} = (A_{ij})$, $\mathscr{B} = (B_{ij})$, the interval matrix addition and product are formally defined by

$$\mathscr{A} + \mathscr{B} = (A_{ij} + B_{ij}),$$

$$\mathscr{A} \cdot \mathscr{B} = \left(\sum_s A_{is} B_{sj} \right),$$

respectively, where the binary operations on the set of intervals have their usual meanings (see [1]). The powers $\mathscr{A}^k$ of $\mathscr{A}$ are defined by

$$\mathscr{A}^k = (\mathscr{A}^{k-1}) \cdot \mathscr{A}, \quad k = 2, 3, \ldots$$

As noted by Mayer, the product of interval matrices is not associative in general. Therefore, $(\mathscr{A}^{k-1}) \cdot \mathscr{A}$ may not equal to $\mathscr{A} \cdot (\mathscr{A}^{k-1})$.

For a real interval $[\underline{a}, \overline{a}]$, the width $d([\underline{a}, \overline{a}])$ and the absolute value $|[\underline{a}, \overline{a}]|$ are defined by

$$d[\underline{a}, \overline{a}] = \overline{a} - \underline{a}, \; |[\underline{a}, \overline{a}]| = \max\{|\underline{a}|, |\overline{a}|\},$$

respectively. For an $n \times n$ interval matrix $\mathscr{A}$, we define the real matrices

$$d(\mathscr{A}) = (d(A_{ij})) \quad \text{and} \quad |\mathscr{A}| = (|A_{ij}|).$$

On the set $M_n(\mathbb{R})$ of real $n \times n$ matrices (a_{ij}) we introduce the usual partial ordering $\leqslant$ by defining $(a_{ij}) \leqslant (b_{ij})$ iff $a_{ij} \leqslant b_{ij}$ for all $1 \leqslant i, j \leqslant n$.

The following important properties of $d(\mathscr{A})$ and $|\mathscr{A}|$ can be found in [2]:

$$d(\mathscr{A}) \geqslant 0, \quad |\mathscr{A}| \geqslant 0,$$

$$d(\mathscr{A}) = 0 \Leftrightarrow \mathscr{A} \text{ is a point matrix},$$

$$|\mathscr{A}| = 0 \Leftrightarrow \mathscr{A} = 0,$$

$$d(\mathscr{A} \pm \mathscr{B}) = d(\mathscr{A}) + d(\mathscr{B}),$$

$$|\mathscr{A} \pm \mathscr{B}| \leqslant |\mathscr{A}| + |\mathscr{B}|,$$

$$d(\mathscr{A})|\mathscr{B}| \leqslant d(\mathscr{A}B) \leqslant d(\mathscr{A})|\mathscr{B}| + |\mathscr{A}|d(\mathscr{B}),$$

$$|\mathscr{A}B| \leqslant |\mathscr{A}||\mathscr{B}|,$$

$$|\mathscr{A}^k| \leqslant |\mathscr{A}|^k.$$

Definition 1 [2]. Let $\mathscr{A} = (A_{ij})$ be an $n \times n$ interval matrix. We say that the *j*th column of $\mathscr{A}$ has property $(*)$ if there exists a power $\mathscr{A}^m$ containing in the same *j*th column at lesast one interval not degenerated to a point interval.

Definition 2 [2]. Let $\mathscr{A} = (A_{ij})$ be an $n \times n$ interval matrix. Construct the real matrix $\tilde{\mathscr{A}} = (a_{ij})$ by

$$a_{ij} = \begin{cases} |A_{ij}| & \text{if the } j\text{th column of } \mathscr{A} \text{ has property } (*), \\ A_{ij} & \text{otherwise.} \end{cases}$$

Lemma 1 [2]. *Let $\mathscr{A} = (A_{ij})$ be an $n \times n$ interval matrix. If the sequence $\{\mathscr{A}^l : l \in \mathbb{N}\}$ converges to the zero matrix and the jth column of $\mathscr{A}$ has property $(*)$, then the jth row of the sequence $\{|\mathscr{A}|^l : l \in \mathbb{N}\}$ converges to the zero vector 0^{T}.*

Lemma 2. *Let $\mathscr{A} = (A_{ij})$ be an $n \times n$ interval matrix. If there exists a nonempty index subset I of $\{1, 2, \ldots, n\}$ such that*

$$\begin{cases} \mathscr{A}_j \text{ does not have property } (*)\ \forall j \in I, \\ \mathscr{A}_j \text{ has property } (*)\ \forall j \in I^c, \end{cases}$$

where $|I| = k < n$ and $I^c := \{1, 2, \ldots, n\} \setminus I$. Then there exists a permutation matrix P such that

$$P^{-1}\mathscr{A}P = \begin{pmatrix} \mathscr{A}_{(2)} & \mathscr{A}_{(3)} \\ 0 & \mathscr{A}_{(1)} \end{pmatrix},$$

where $\mathscr{A}_{(2)}$ is a $k \times k$ point matrix and each column located in $\binom{\mathscr{A}_{(3)}}{\mathscr{A}_{(1)}}$ has property $(*)$.

Proof. Choose a permutation matrix P so that

$$\begin{cases} (P^{-1}\mathscr{A}P)_j \text{ does not have property } (*)\ \forall j = 1, 2, \ldots, k, \\ (P^{-1}\mathscr{A}P)_j \text{ has property } (*)\ \forall j = k+1, \ldots, n. \end{cases}$$

We now demonstrate that each *rs* entry of $P^{-1}\mathscr{A}P$ satisfies

$$(P^{-1}\mathscr{A}P)_{rs} = 0 \quad \text{for } k+1 \leqslant r \leqslant n \text{ and } 1 \leqslant s \leqslant k.$$

Suppose to the contrary that

$$(P^{-1}\mathscr{A}P)_{rs} \neq 0 \quad \text{for some } k+1 \leqslant r \leqslant n \text{ and some } 1 \leqslant s \leqslant k.$$

Since the *r*th column of $P^{-1}\mathscr{A}P$ has property $(*)$, there exists a positive integer m such that the width

$$d(P^{-1}\mathscr{A}^m P)_{\alpha r} \neq 0 \quad \text{for some } 1 \leqslant \alpha \leqslant n.$$

Hence

$$d(P^{-1}\mathscr{A}^{m+1}P)_{\alpha s} = d[(P^{-1}\mathscr{A}^m P)(P^{-1}\mathscr{A}P)]_{\alpha s} \neq 0.$$

Hence, the sth column of $P^{-1}\mathscr{A}P$ has property $(*)$, a contradiction. Therefore $P^{-1}\mathscr{A}P$ can be represented in the block interval matrix form as

$$P^{-1}\mathscr{A}P = \begin{pmatrix} \mathscr{A}_{(2)} & \mathscr{A}_{(3)} \\ 0 & \mathscr{A}_{(1)} \end{pmatrix},$$

where $\mathscr{A}_{(2)}$ is a $k \times k$ point matrix and each column located in $\binom{\mathscr{A}_{(3)}}{\mathscr{A}_{(1)}}$ has property $(*)$. $\square$

Prior to Mayer's theorem, we recall some well-known results for an $n \times n$ complex matrix A (see, for example, [3, pp. 12, 73]). $A^k \to 0$ as $k \to \infty$ iff $\rho(A) < 1$. Furthermore, $A^k \to 0$ as $k \to \infty$ iff there exists a norm $\|\cdot\|$ on $\mathbb{C}^n$ such that $\|A\| < 1$.

We are ready to provide a new proof for Mayer's theorem concerning the convergence of powers of an interval matrix, to which we now turn.

Theorem 1 [2]. *Let $\mathscr{A} = (A_{ij})$ be an $n \times n$ interval matrix. Then the sequence $\{\mathscr{A}^l : l \in \mathbb{N}\}$ converges to 0 if and only if $\rho(\tilde{\mathscr{A}}) < 1$.*

Proof. We distinguish three cases:

Case 1. If all columns of $\mathscr{A}$ do not have property $(*)$, then $\mathscr{A}$ is a point matrix and $\mathscr{A} = \tilde{\mathscr{A}}$. Moreover,

$$\mathscr{A}^k \to 0 \text{ as } k \to \infty \quad \text{if and only if } \rho(\tilde{\mathscr{A}}) < 1.$$

Case 2. If each column of $\mathscr{A}$ has property $(*)$, then $\tilde{\mathscr{A}} = |\mathscr{A}|$. For $l = 1, 2, \ldots,$ we have

$$|\mathscr{A}^l| \leqslant |\mathscr{A}|^l = \tilde{\mathscr{A}}^l.$$

"$\Rightarrow$" Assume $\lim_{l\to\infty} \mathscr{A}^l = 0$. By Lemma 1 we have $|\mathscr{A}|^l \to 0$ as $l \to \infty$. It follows that $\rho(\tilde{\mathscr{A}}) = \rho(|\mathscr{A}|) < 1$.
"$\Leftarrow$" Assume $\rho(\tilde{\mathscr{A}}) < 1$. Then $\rho(\tilde{\mathscr{A}}) = \rho(|\mathscr{A}|) < 1$. It follows that $\lim_{l\to\infty} |\mathscr{A}|^l = 0$. Since $|\mathscr{A}^k| \leqslant |\mathscr{A}|^k$ for each k, we then have $\mathscr{A}^k \to 0$ as $k \to \infty$.

Case 3. If there exists a nonempty index subset I of $\{1, 2, \ldots, n\}$ such that

$$\begin{cases} \mathscr{A}_j \text{ does not have property } (*) \ \forall\, j \in I, \\ \mathscr{A}_j \text{ has property } (*) \ \forall\, j \in I^c, \end{cases}$$

where $|I| = k < n$. By Lemma 2 there exists a permutation matrix P such that

$$P^{-1}\mathscr{A}P = \mathscr{D}_{\mathscr{A}} + \mathscr{N}_{\mathscr{A}},$$

where

$$\mathscr{D}_{\mathscr{A}} = \begin{pmatrix} \mathscr{A}_{(2)} & 0 \\ 0 & \mathscr{A}_{(1)} \end{pmatrix} \quad \text{and} \quad \mathscr{N}_{\mathscr{A}} = \begin{pmatrix} 0 & \mathscr{A}_{(3)} \\ 0 & 0 \end{pmatrix}.$$

Here all columns of $\binom{\mathscr{A}_{(2)}}{0}$ do not have property $(*)$ while each column located in $\binom{\mathscr{A}_{(3)}}{\mathscr{A}_{(1)}}$ has property $(*)$. Thus

$$(\widetilde{P^{-1}\mathscr{A}P}) = P^{-1}\tilde{\mathscr{A}}P = \mathscr{D}_{\tilde{\mathscr{A}}} + \mathscr{N}_{\tilde{\mathscr{A}}},$$

where

$$\mathscr{D}_{\tilde{\mathscr{A}}} = \begin{pmatrix} \mathscr{A}_{(2)} & 0 \\ 0 & |\mathscr{A}_{(1)}| \end{pmatrix} \quad \text{and} \quad \mathscr{N}_{\tilde{\mathscr{A}}} = \begin{pmatrix} 0 & |\mathscr{A}_{(3)}| \\ 0 & 0 \end{pmatrix}.$$

"$\Leftarrow$" Assume that $\rho(\tilde{\mathscr{A}}) < 1$. Then $\rho(P^{-1}\tilde{\mathscr{A}}P) < 1$. It follows that there exists a norm $\|\cdot\|$ on $\mathbb{R}^n$ such that $\|P^{-1}\tilde{\mathscr{A}}P\| < 1$. For[1] $k = 1, 2, \ldots,$

$$\begin{aligned} P^{-1}\mathscr{A}^k P &= \underbrace{P^{-1}\mathscr{A}P \cdots P^{-1}\mathscr{A}P}_{k\text{-times}} \\ &= \underbrace{[\mathscr{D}_{\mathscr{A}} + \mathscr{N}_{\mathscr{A}}] \cdots [\mathscr{D}_{\mathscr{A}} + \mathscr{N}_{\mathscr{A}}]}_{k\text{-times}} \\ &= \underbrace{\mathscr{D}_{\mathscr{A}} \cdots \mathscr{D}_{\mathscr{A}}}_{k\text{-matrices}} + \cdots + \underbrace{\mathscr{N}_{\mathscr{A}} \cdots \mathscr{N}_{\mathscr{A}}}_{k\text{-matrices}}. \end{aligned}$$

Note that the terms which contain at least two nilpotent factors must be zero. Thus

$$P^{-1}\mathscr{A}^k P = (\mathscr{D}_{\mathscr{A}})^k + \Omega^1_{\mathscr{A}} + \cdots + \Omega^k_{\mathscr{A}},$$

where $\Omega^i_{\mathscr{A}}$ denotes

$$\underbrace{\mathscr{D}_{\mathscr{A}} \cdots \mathscr{D}_{\mathscr{A}}}_{(i-1)\text{-matrices}} \; \mathscr{N}_{\mathscr{A}} \; \underbrace{\mathscr{D}_{\mathscr{A}} \cdots \mathscr{D}_{\mathscr{A}}}_{(k-i)\text{-matrices}}$$

with $\mathscr{N}_{\mathscr{A}}$ appearing in the ith place of the product. Let $\|\cdot\|_{\mathrm{F}}$ denote the Frobenius norm.

We then have

$$\begin{aligned} &\| |P^{-1}\mathscr{A}^k P| \|_{\mathrm{F}} \\ &\quad \leqslant \| |\mathscr{D}^k_{\mathscr{A}}| \|_{\mathrm{F}} + \| \left(\sum_{i=1}^{k} |\Omega^i_{\mathscr{A}}| \right) \|_{\mathrm{F}} \\ &\quad \leqslant \| |\mathscr{D}^k_{\mathscr{A}}| \|_{\mathrm{F}} + \sum_{i=1}^{k} \| | \underbrace{\mathscr{D}_{\mathscr{A}} \cdots \mathscr{D}_{\mathscr{A}}}_{(i-1)\text{-times}} | \|_{\mathrm{F}} \| |\mathscr{N}_{\mathscr{A}}| \|_{\mathrm{F}} \| | \underbrace{\mathscr{D}_{\mathscr{A}} \cdots \mathscr{D}_{\mathscr{A}}}_{(k-i)\text{-times}} | \|_{\mathrm{F}} \\ &\quad \leqslant \|\mathscr{D}^k_{\tilde{\mathscr{A}}}\|_{\mathrm{F}} + \sum_{i=1}^{k} \|\mathscr{D}_{\tilde{\mathscr{A}}} \cdots \mathscr{D}_{\tilde{\mathscr{A}}}\|_{\mathrm{F}} \|\mathscr{N}_{\tilde{\mathscr{A}}}\|_{\mathrm{F}} \|\mathscr{D}_{\tilde{\mathscr{A}}} \cdots \mathscr{D}_{\tilde{\mathscr{A}}}\|_{\mathrm{F}} \end{aligned}$$

[1] To ease our notation, we omit the parentheses in products of interval matrices. Namely, $\mathscr{A} \cdot \mathscr{B} \cdot \mathscr{C}$, $\mathscr{A} \cdot \mathscr{B} \cdot \mathscr{C} \cdot \mathscr{D}$, etc., instead of $(\mathscr{A} \cdot \mathscr{B}) \cdot \mathscr{C}$, $((\mathscr{A} \cdot \mathscr{B}) \cdot \mathscr{C}) \cdot \mathscr{D}$, etc.

$$
\begin{aligned}
&\leqslant \|(P^{-1}\tilde{\mathscr{A}}P)^k\|_{\mathrm{F}} + \sum_{i=1}^{k} \|(P^{-1}\tilde{\mathscr{A}}P)\cdots(P^{-1}\tilde{\mathscr{A}}P)\|_{\mathrm{F}} \\
&\quad \times \|\mathscr{N}_{\tilde{\mathscr{A}}}\|_{\mathrm{F}} \|(P^{-1}\tilde{\mathscr{A}}P)\cdots(P^{-1}\tilde{\mathscr{A}}P)\|_{\mathrm{F}} \\
&\leqslant c_0\|(P^{-1}\tilde{\mathscr{A}}P)^k\| + c_0^2 \sum_{i=1}^{k} \|(P^{-1}\tilde{\mathscr{A}}P)\cdots(P^{-1}\tilde{\mathscr{A}}P)\| \\
&\quad \times \|\mathscr{N}_{\tilde{\mathscr{A}}}\|_{\mathrm{F}} \|(P^{-1}\tilde{\mathscr{A}}P)\cdots(P^{-1}\tilde{\mathscr{A}}P)\| \\
&\leqslant c_0\|(P^{-1}\tilde{\mathscr{A}}P)\|^k + c_0^2 k \|\mathscr{N}_{\tilde{\mathscr{A}}}\|_{\mathrm{F}} \|(P^{-1}\tilde{\mathscr{A}}P)\|^{k-1} \to 0 \text{ as } k \to \infty,
\end{aligned}
$$

where the constant c_0 occurs because of changing norms. Hence $\mathscr{A}^k \to 0$ as $k \to \infty$.

"$\Rightarrow$" Assume $\mathscr{A}^k \to 0$ as $k \to \infty$. Then

$$P^{-1}\mathscr{A}^k P = \underbrace{P^{-1}\mathscr{A}P \cdots P^{-1}\mathscr{A}P}_{k\text{-times}} \to 0 \quad \text{as } k \to \infty.$$

It follows that $\mathscr{A}_{(2)}^k \to 0$ as $k \to \infty$, where $\mathscr{A}_{(2)}^k$ is defined analogously to $\mathscr{A}_{(2)}$. By Lemma 1 we have $|\mathscr{A}_{(1)}|^k \to 0$ as $k \to \infty$. Therefore

$$\rho(\tilde{\mathscr{A}}) = \rho(P^{-1}\tilde{\mathscr{A}}P) = \max\{\rho(\mathscr{A}_{(2)}), \rho(|\mathscr{A}_{(1)}|)\} < 1.$$

This completes the proof. □

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